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J. Math. Anal. Appl. 275 (2002) 900–912

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.academicpress.com

Polynomial characterization of $\mathcal{L}_\infty$-spaces

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Received 10 July 2001

Submitted by J. Diestel

Abstract

It is shown that, given an index m, a Banach space E is an $\mathcal{L}_\infty$-space if and only if every 1-dominated m-homogeneous polynomial on E is integral. This extends a result for linear operators due to Stegall and Retherford.

Keywords: 1-dominated polynomials; Integral polynomials; $\mathcal{L}_\infty$-space

1. Introduction

The absolutely r-summing and the integral operators constitute a very important chapter in the theory of (linear bounded) operators between Banach spaces.

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1 This work was performed during a six-month visit of the author to the Universidad Complutense de Madrid thanks to a grant of the Italian CNR.

2 The author was supported in part by MURST (Italy).

3 The author was supported in part by Dirección General de Investigación, BFM 2000-0609 (Spain).

PII: S0022-247X(02)00461-4

In [15] and [14] there are results relating geometric properties of Banach spaces with the coincidence of the above mentioned classes of operators. In [15] it is shown in particular that a Banach space E is an $\mathcal{L}_\infty$-space if and only if every absolutely summing operator on E is integral.

In Section 2 of the present paper, we introduce and study the integral multilinear mappings and polynomials, extending the definition given by Pietsch [12] for multilinear functionals, and we show, for instance, that a polynomial is integral if and only if its associated symmetric multilinear mapping is integral, if and only if its linearization is an integral operator. We also prove that the composition of an integral operator with a polynomial is integral.

The natural extension of the definition of absolutely r-summing operators to the setting of polynomials yields the r-dominated polynomials (see definition below), which have been studied, e.g., in [9,10].

In Section 3, we obtain our main result: given an integer m, a Banach space E is an $\mathcal{L}_\infty$-space if and only if every 1-dominated m-homogeneous polynomial on E is integral.

Throughout, E and F denote Banach spaces, E^* is the dual of E, and B_E stands for its closed unit ball. By $\mathbb{N}$ we represent the set of all natural numbers and by $\mathbb{K}$ the scalar field (real or complex). Given $m \in \mathbb{N}$, we denote by $\mathcal{P}(^mE, F)$ the space of all m-homogeneous (continuous) polynomials from E into F endowed with the supremum norm. If $E_1, \dots, E_m$ are Banach spaces, $\mathcal{L}(^mE_1, \dots, E_m; F)$ denotes the space of all m-linear (continuous) mappings from $E_1 \times \cdots \times E_m$ into F, also endowed with the supremum norm. If $E_1 = \cdots = E_m = E$, that space is represented by $\mathcal{L}(^mE, F)$. Recall that to each $P \in \mathcal{P}(^mE, F)$ we can associate a unique symmetric $\widehat{P} \in \mathcal{L}(^mE, F)$ so that

$$P(x) = \widehat{P}\big(x, \overset{(m)}{\dots}, x\big) \quad (x \in E).$$

The norms are related by the inequalities [11, Theorem 2.2]:

$$\|P\| \leqslant \|\widehat{P}\| \leqslant \frac{m^m}{m!}\|P\|. \tag{1}$$

For the general theory of polynomials on Banach spaces, we refer to [5] and [11].

Given $1 \leqslant r < \infty$, a polynomial $P \in \mathcal{P}(^mE, F)$ is *r-dominated* if there exists a constant $k > 0$ such that, for all $n \in \mathbb{N}$ and $(x_i)_{i=1}^n \subset E$, we have

$$\left(\sum_{i=1}^n \|P(x_i)\|^{r/m}\right)^{m/r} \leqslant k \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^n |x^*(x_i)|^r\right)^{m/r}.$$

We denote by $\mathcal{P}_{\mathrm{as}}(^mE, F)$ the space of all 1-dominated polynomials from E into F.

We use the notation $\bigotimes^m E := E \otimes \overset{(m)}{\cdots} \otimes E$ for the m-fold tensor product of E, $\bigotimes_\epsilon^m E := E \otimes_\epsilon \overset{(m)}{\cdots} \otimes_\epsilon E$ for the m-fold injective tensor product of E, and $\bigotimes_\pi^m E$ for the m-fold projective tensor product of E (see [4] for the theory of tensor products). By $\bigotimes_s^m E := E \otimes_s \overset{(m)}{\cdots} \otimes_s E$ we denote the m-fold symmetric tensor product of E, i.e., the set of all elements $u \in \bigotimes^m E$ of the form

$$u = \sum_{j=1}^{n} \lambda_j x_j \otimes \overset{(m)}{\cdots} \otimes x_j \quad (n \in \mathbb{N},\ \lambda_j \in \mathbb{K},\ x_j \in E,\ 1 \leqslant j \leqslant n).$$

By $\bigotimes_{\epsilon,s}^m E$ we denote the closure of $\bigotimes_s^m E$ in $\bigotimes_\epsilon^m E$. Analogously, $\bigotimes_{\pi,s}^m E$ is the closure of $\bigotimes_s^m E$ in $\bigotimes_\pi^m E$. For symmetric tensor products, we refer to [6].

We denote by $\mathcal{AS}(E, F)$ the space of all absolutely summing operators from E into F and by $\mathcal{I}(E, F)$ the space of all integral operators. The definitions may be seen in [4] and [3].

We shall use the fact [4, p. 232] that an operator $T : E \to F$ is integral if and only if the functional $\widetilde{T} : E \otimes_\epsilon F^* \to \mathbb{K}$, given by $\widetilde{T}(x \otimes f^*) = \langle T(x), f^* \rangle$ for $x \in E$, $f^* \in F^*$, is well-defined and continuous.

The *Banach–Mazur distance* $d(E, F)$ between two isomorphic Banach spaces E and F is defined by $\inf(\|T\|\|T^{-1}\|)$ where the infimum is taken over all isomorphisms T from E onto F. Recall that a Banach space E is an $\mathcal{L}_\infty$*-space* [8] if there is $\lambda \geqslant 1$ such that every finite dimensional subspace of E is contained in another subspace N with $d(N, \ell_\infty^n) \leqslant \lambda$ for some integer n. Every $C(K)$ space is an $\mathcal{L}_\infty$-space.

2. Integral polynomials and multilinear mappings

In this section, we introduce and study the integral multilinear mappings and polynomials proving, among other results, that a polynomial is integral if and only if its associated symmetric multilinear mapping is integral, if and only if its linearization is integral. It is also proved that, given $Q \in \mathcal{P}(^m G, F)$ and $T \in \mathcal{I}(E, G)$, the composition $Q \circ T \in \mathcal{P}(^m E, F)$ is integral.

Definition 2.1. We say that $T \in \mathcal{L}(^m E_1, \dots, E_m; F)$ is *integral* if there exists a constant $C \geqslant 0$ such that for every $n \in \mathbb{N}$, and all families $(x_{1j})_{j=1}^n \subset E_1, \dots, (x_{mj})_{j=1}^n \subset E_m$ and $(f_j^*)_{j=1}^n \subset F^*$, we have

$$\left| \sum_{j=1}^{n} \langle T(x_{1j}, \dots, x_{mj}), f_j^* \rangle \right| \leqslant C \sup_{\substack{x_k^* \in B_{E_k^*} \\ k=1,\dots,m}} \left\| \sum_{j=1}^{n} x_1^*(x_{1j}) \dots x_m^*(x_{mj}) f_j^* \right\|_{F^*}.$$

In the case $F = \mathbb{K}$, this definition was given in [12] (see also [7]). Easily, for $m = 1$, we obtain the (Grothendieck) integral operators [4, p. 232].

If $T \in \mathcal{L}(^m E_1, \ldots, E_m; F)$, we denote by $\overline{T}$ the linearization of T, that is the linear map $\overline{T} : E_1 \otimes \cdots \otimes E_m \to F$ given by

$$\overline{T}\left(\sum_{j=1}^{n} x_{1j} \otimes \cdots \otimes x_{mj}\right) = \sum_{j=1}^{n} T(x_{1j}, \ldots, x_{mj})$$

for all $x_{kj} \in E_k$ $(1 \leqslant k \leqslant m, 1 \leqslant j \leqslant n)$.

Proposition 2.2. *The mapping $T \in \mathcal{L}(^m E_1, \ldots, E_m; F)$ is integral if and only if $\overline{T} : E_1 \otimes_\epsilon \cdots \otimes_\epsilon E_m \to F$ is well-defined and integral.*

Proof. Suppose that T is integral. Easily, $\overline{T}$ is continuous on the uncompleted tensor product endowed with the ϵ-norm and so it is well-defined on $E_1 \otimes_\epsilon \cdots \otimes_\epsilon E_m$. On the other hand, $\overline{T}$ is integral if and only if the linear functional

$$\tilde{\overline{T}} : (E_1 \otimes_\epsilon \cdots \otimes_\epsilon E_m) \otimes F^* \to \mathbb{K},$$

defined by

$$\tilde{\overline{T}}\left(\sum_{i=1}^{n} u_i \otimes f_i^*\right) = \sum_{i=1}^{n} \langle \overline{T}(u_i), f_i^* \rangle$$

for all $u_i \in E_1 \otimes_\epsilon \cdots \otimes_\epsilon E_m$ and $f_i^* \in F^*$ $(1 \leqslant i \leqslant n)$, is continuous with respect to the injective norm on $(E_1 \otimes_\epsilon \cdots \otimes_\epsilon E_m) \otimes F^*$. We observe that, if

$$u_i = \sum_{j=1}^{r_i} x_{1j}^{(i)} \otimes \cdots \otimes x_{mj}^{(i)}, \quad \text{with } x_{kj}^{(i)} \in E_k$$
$$(1 \leqslant k \leqslant m,\ 1 \leqslant j \leqslant r_i,\ 1 \leqslant i \leqslant n),$$

then

$$\begin{aligned}\left|\tilde{\overline{T}}\left(\sum_{i=1}^{n} u_i \otimes f_i^*\right)\right| &= \left|\sum_{i=1}^{n} \left\langle \overline{T}\left(\sum_{j=1}^{r_i} x_{1j}^{(i)} \otimes \cdots \otimes x_{mj}^{(i)}\right), f_i^* \right\rangle\right| \\ &= \left|\sum_{i=1}^{n} \left\langle \sum_{j=1}^{r_i} T\left(x_{1j}^{(i)}, \ldots, x_{mj}^{(i)}\right), f_i^* \right\rangle\right| \\ &= \left|\sum_{i=1}^{n} \sum_{j=1}^{r_i} \left\langle T\left(x_{1j}^{(i)}, \ldots, x_{mj}^{(i)}\right), f_i^* \right\rangle\right|.\end{aligned}$$

Now, using Definition 2.1 for the families

$$\left(x_{1j}^{(i)}\right)_{\substack{j=1,\ldots,r_i \\ i=1,\ldots,n}}, \ldots, \left(x_{mj}^{(i)}\right)_{\substack{j=1,\ldots,r_i \\ i=1,\ldots,n}}, \qquad \left(f_{ij}^*\right)_{\substack{j=1,\ldots,r_i \\ i=1,\ldots,n}},$$

where $f_{ij}^* = f_i^*$ $(j = 1, \ldots, r_i,\ i = 1, \ldots, n)$, we obtain

$$\begin{aligned}
\left|\tilde{\tilde{T}}\left(\sum_{i=1}^{n} u_i \otimes f_i^*\right)\right| &\leqslant C \sup_{\substack{x_k^* \in B_{E_k^*} \\ k=1,\dots,m}} \left\| \sum_{i=1}^{n} \sum_{j=1}^{r_i} x_1^*\big(x_{1j}^{(i)}\big) \dots x_m^*\big(x_{mj}^{(i)}\big) f_{ij}^* \right\|_{F^*} \\
&= C \sup_{\substack{x_k^* \in B_{E_k^*} \\ k=1,\dots,m}} \left\| \sum_{i=1}^{n} (x_1^* \otimes \dots \otimes x_m^*) \left(\sum_{j=1}^{r_i} x_{1j}^{(i)} \otimes \dots \otimes x_{mj}^{(i)} \right) f_i^* \right\|_{F^*} \\
&\leqslant C \sup \left\{ \left| \sum_{i=1}^{n} \Phi(u_i) g\big(f_i^*\big) \right| : \Phi \in (E_1 \otimes_\epsilon \dots \otimes_\epsilon E_m)^*, \right. \\
&\qquad\qquad \left. \|\Phi\| \leqslant 1,\ g \in B_{F^{**}} \right\} \\
&= C \left\| \sum_{i=1}^{n} u_i \otimes f_i^* \right\|_{\epsilon}.
\end{aligned}$$

Conversely, suppose that $\tilde{\tilde{T}}$ is continuous with respect to the ϵ-norm on $(E_1 \otimes_\epsilon \dots \otimes_\epsilon E_m) \otimes F^*$. Let $n \in \mathbb{N}$, $(x_{1i})_{i=1}^n \subset E_1, \dots, (x_{mi})_{i=1}^n \subset E_m$ and $(f_i^*)_{i=1}^n \subset F^*$ be fixed. We have

$$\begin{aligned}
\left| \sum_{i=1}^{n} \big\langle T(x_{1i}, \dots, x_{mi}), f_i^* \big\rangle \right| &= \left| \sum_{i=1}^{n} \tilde{\tilde{T}}\big[(x_{1i} \otimes \dots \otimes x_{mi}) \otimes f_i^*\big] \right| \\
&= \left| \tilde{\tilde{T}}\left[\sum_{i=1}^{n} (x_{1i} \otimes \dots \otimes x_{mi}) \otimes f_i^* \right] \right| \\
&\leqslant C \sup_{\substack{x_k^* \in B_{E_k^*} \\ k=1,\dots,m \\ g \in B_{F^{**}}}} \left| \sum_{i=1}^{n} x_i^*(x_{1i}) \dots x_m^*(x_{mi}) g\big(f_i^*\big) \right| \\
&= C \sup_{\substack{x_k^* \in B_{E_k^*} \\ k=1,\dots,m}} \left\| \sum_{i=1}^{n} x_1^*(x_{1i}) \dots x_m^*(x_{mi}) f_i^* \right\|_{F^*},
\end{aligned}$$

where we have used the associative property of the injective tensor product. Therefore, T is integral. □

In [16], the author introduces the integral multilinear mappings as those satisfying a certain integral expression. Proposition 2.2 implies that his definition is equivalent to ours.

We now introduce the integral polynomials.

Definition 2.3. We say that $P \in \mathcal{P}(^mE, F)$ is *integral* if there exists a constant $C \geqslant 0$ such that for every $n \in \mathbb{N}$, and all families $(x_i)_{i=1}^n \subseteq E$ and $(f_i^*)_{i=1}^n \subset F^*$, we have

$$\left|\sum_{i=1}^n \langle P(x_i), f_i^*\rangle\right| \leqslant C \sup_{x^* \in B_{E^*}} \left\|\sum_{i=1}^n \left[x^*(x_i)\right]^m f_i^*\right\|_{F^*}.$$

We use the symbol $\mathcal{P}_{\mathrm{I}}(^mE, F)$ to denote the space of all m-homogeneous integral polynomials from E into F.

The following proposition relates integral m-homogeneous polynomials with m-linear integral operators.

Proposition 2.4. *The polynomial $P \in \mathcal{P}(^mE, F)$ is integral if and only if $\widehat{P} \in \mathcal{L}(^mE, F)$ is integral.*

Proof. We suppose that $\widehat{P}$ is integral. Fixed $n \in \mathbb{N}$, $x_i \in E$, and $f_i^* \in F^*$ $(i = 1, \dots, n)$, we have

$$\begin{aligned}\left|\sum_{i=1}^n \langle P(x_i), f_i^*\rangle\right| &= \left|\sum_{i=1}^n \left\langle \widehat{P}\left(x_i, \overset{(m)}{\dots}, x_i\right), f_i^*\right\rangle\right| \\ &\leqslant C \sup_{\substack{x_k^* \in B_{E^*} \\ k=1,\dots,m}} \left\|\sum_{i=1}^n x_1^*(x_i)\dots x_m^*(x_i) f_i^*\right\|_{F^*}.\end{aligned}$$

This supremum is the norm of the symmetric m-linear mapping

$$(x_1^*, \dots, x_m^*) \in E^* \times \overset{(m)}{\cdots} \times E^* \mapsto \sum_{i=1}^n x_1^*(x_i)\dots x_m^*(x_i) f_i^* \in F^*.$$

Considering its associated polynomial and using (1), we get

$$\left|\sum_{i=1}^n \langle P(x_i), f_i^*\rangle\right| \leqslant C \frac{m^m}{m!} \sup_{x^* \in B_{E^*}} \left\|\sum_{i=1}^n \left[x^*(x_i)\right]^m f_i^*\right\|_{F^*}.$$

So P is integral.

Conversely, let P be integral. Fix $n \in \mathbb{N}$, $(x_{ki})_{i=1}^n \subset E$ $(1 \leqslant k \leqslant m)$ and $(f_i^*)_{i=1}^n \subset F^*$. Using the polarization formula [11, Theorem 1.10], we have

$$\begin{aligned}&\left|\sum_{i=1}^n \left\langle \widehat{P}(x_{1i}, \dots, x_{mi}), f_i^*\right\rangle\right| \\ &\quad = \left|\sum_{i=1}^n \left\langle \frac{1}{m!2^m} \sum_{\substack{\epsilon_j = \pm 1 \\ 1 \leqslant j \leqslant m}} \epsilon_1 \dots \epsilon_m P(\epsilon_1 x_{1i} + \dots + \epsilon_m x_{mi}), f_i^*\right\rangle\right|\end{aligned}$$

$$= \frac{1}{m!2^m} \left| \sum_{i=1}^{n} \sum_{\substack{\epsilon_j=\pm 1 \\ 1\leqslant j\leqslant m}} \left\langle P(\epsilon_1 x_{1i} + \cdots + \epsilon_m x_{mi}), \epsilon_1 \ldots \epsilon_m f_i^* \right\rangle \right|.$$

Now we can apply Definition 2.3 to the families

$$(\epsilon_1 x_{1i} + \cdots + \epsilon_m x_{mi})_{\substack{i=1,\ldots,n \\ \epsilon_j=\pm 1 \\ j=1,\ldots,m}} \quad \text{and} \quad \left(\epsilon_1 \ldots \epsilon_m f_i^*\right)_{\substack{i=1,\ldots,n \\ \epsilon_j=\pm 1 \\ j=1,\ldots,m}}$$

obtaining

$$\left| \sum_{i=1}^{n} \left\langle \widehat{P}(x_{1i}, \ldots, x_{mi}), f_i^* \right\rangle \right|$$
$$\leqslant \frac{1}{m!2^m} C \sup_{x^* \in B_{E^*}} \left\| \sum_{i=1}^{n} \sum_{\substack{\epsilon_j=\pm 1 \\ 1\leqslant j\leqslant m}} \left[x^*(\epsilon_1 x_{1i} + \cdots + \epsilon_m x_{mi})\right]^m \epsilon_1 \cdots \epsilon_m f_i^* \right\|_{F^*}.$$

By the polarization formula, for every $x^* \in B_{E^*}$ and every $i \in \{1, \ldots, n\}$, we have

$$\frac{1}{2^m m!} \sum_{\substack{\epsilon_j=\pm 1 \\ 1\leqslant j\leqslant m}} \epsilon_1 \ldots \epsilon_m \left[x^*(\epsilon_1 x_{1i} + \cdots + \epsilon_m x_{mi})\right]^m$$
$$= x^*(x_{1i}) x^*(x_{2i}) \ldots x^*(x_{mi}).$$

So, we obtain

$$\left| \sum_{i=1}^{n} \left\langle \widehat{P}(x_{1i}, \ldots, x_{mi}), f_i^* \right\rangle \right| \leqslant C \sup_{x^* \in B_{E^*}} \left\| \sum_{i=1}^{n} x^*(x_{1i}) \ldots x^*(x_{mi}) f_i^* \right\|_{F^*}$$
$$\leqslant C \sup_{\substack{x_k^* \in B_{E^*} \\ k=1,\ldots,m}} \left\| \sum_{i=1}^{n} x_1^*(x_{1i}) \ldots x_m^*(x_{mi}) f_i^* \right\|_{F^*},$$

and hence $\widehat{P}$ is integral. □

If $P \in \mathcal{P}(^m E, F)$, we define its linearization $\overline{P} : \bigotimes_s^m E \to F$ by

$$\overline{P}\left(\sum_{i=1}^{n} \lambda_i x_i \otimes \overset{(m)}{\cdots} \otimes x_i \right) = \sum_{i=1}^{n} \lambda_i P(x_i)$$

for all $\lambda_i \in \mathbb{K}$, $x_i \in E$ $(1 \leqslant i \leqslant n)$.

Proposition 2.5. *The polynomial $P \in \mathcal{P}(^m E, F)$ is integral if and only if the linear map $\overline{P} : \bigotimes_{\epsilon,s}^m E \to F$ is well-defined and integral.*

Proof. Suppose P is integral. Since $\overline{P}$ is the restriction to $\bigotimes_{\epsilon,s}^m E$ of $\widehat{\overline{P}}$ (the linearization of $\widehat{P}$), using Propositions 2.4 and 2.2, we obtain that $\overline{P}$ is integral.

Conversely, suppose that $\overline{P}$ is integral. Then it satisfies Definition 2.1, so there is $C \geqslant 0$ such that, given $n \in \mathbb{N}$, $x_i \in E$, and $f_i^* \in F^*$ $(1 \leqslant i \leqslant n)$, we have

$$\left|\sum_{i=1}^n \langle P(x_i), f_i^*\rangle\right| = \left|\sum_{i=1}^n \langle \overline{P}\big(x_i \otimes \overset{(m)}{\cdots} \otimes x_i\big), f_i^*\rangle\right| \leqslant C \sup_{\substack{\Phi \in (\bigotimes_{\epsilon,s}^m E)^* \\ \|\Phi\| \leqslant 1}} \left\| \sum_{i=1}^n \Phi\big(x_i \otimes \overset{(m)}{\cdots} \otimes x_i\big) f_i^* \right\|_{F^*}.$$

We can view the tensor

$$\sum_{i=1}^n \big(x_i \otimes \overset{(m)}{\cdots} \otimes x_i\big) \otimes f_i^* \in \Big(\bigotimes\nolimits_{\epsilon,s}^m E\Big) \otimes_\epsilon F^*$$

as an operator

$$\Big(\bigotimes\nolimits_{\epsilon,s}^m E\Big)^* \to F^*$$

given by

$$\left[\sum_{i=1}^n \big(x_i \otimes \overset{(m)}{\cdots} \otimes x_i\big) \otimes f_i^*\right](\Phi) = \sum_{i=1}^n \Phi\big(x_i \otimes \overset{(m)}{\cdots} \otimes x_i\big) f_i^*$$

for all $\Phi \in (\bigotimes_{\epsilon,s}^m E)^*$. The tensor and the operator have equal norms. Therefore,

$$\begin{aligned}
\left|\sum_{i=1}^n \langle P(x_i), f_i^*\rangle\right| &\leqslant C \sup_{\substack{\Phi \in (\bigotimes_{\epsilon,s}^m E)^* \\ \|\Phi\| \leqslant 1}} \left\| \left[\sum_{i=1}^n \big(x_i \otimes \overset{(m)}{\cdots} \otimes x_i\big) \otimes f_i^*\right](\Phi) \right\|_{F^*} \\
&= C \left\| \sum_{i=1}^n \big(x_i \otimes \overset{(m)}{\cdots} \otimes x_i\big) \otimes f_i^* \right\|_\epsilon \\
&= C \sup_{\substack{x_j^* \in B_{E^*} \\ j=1,\dots,m \\ \|g\|_{F^{**}} \leqslant 1}} \left| \sum_{i=1}^n x_1^*(x_i) \dots x_m^*(x_i) g(f_i^*) \right| \\
&= C \sup_{\substack{x_j^* \in B_{E^*} \\ j=1,\dots,m}} \left\| \sum_{i=1}^n x_1^*(x_i) \dots x_m^*(x_i) f_i^* \right\|_{F^*} \\
&\leqslant C' \sup_{x^* \in B_{E^*}} \left\| \sum_{i=1}^n \big[x^*(x_i)\big]^m f_i^* \right\|_{F^*}
\end{aligned}$$

as in the proof of Proposition 2.4. Hence, P is integral. □

In the proof of the following proposition, we shall use the fact [2, Theorem 10.7] that an operator $T : E \to F$ is integral if and only if, for every Banach space G, the linear map

$$T \otimes \mathrm{Id}_G : E \otimes_\epsilon G \to F \otimes_\pi G$$

is continuous, where Id_G denotes the identity map on G.

Proposition 2.6. *Let $E_1, \ldots, E_m$, $F_1, \ldots, F_m$, F be Banach spaces. Suppose that $A_i : E_i \to F_i$ $(1 \leqslant i \leqslant m)$ are integral operators and let $B \in \mathcal{L}(^m F_1, \ldots, F_m; F)$. Then the composition $B(A_1, \ldots, A_m) \in \mathcal{L}(^m E_1, \ldots, E_m; F)$ is integral.*

Proof. According to Proposition 2.2, we have to prove that $\overline{B(A_1, \ldots, A_m)}$: $E_1 \otimes_\epsilon \cdots \otimes_\epsilon E_m \to F$ is well-defined and integral. Suppose we have proved that it is continuous on the uncompleted tensor product $E_1 \otimes \cdots \otimes E_m$ endowed with the ϵ-norm and hence that it is well-defined on $E_1 \otimes_\epsilon \cdots \otimes_\epsilon E_m$. We have then to show that it is integral, equivalently, that the functional

$$\left(\overline{B(A_1, \ldots, A_m)}\right)^{\sim} : E_1 \otimes_\epsilon \cdots \otimes_\epsilon E_m \otimes_\epsilon F^* \to \mathbb{K}$$

is well-defined and continuous. Since B is continuous, so are the linear map

$$\overline{B} : F_1 \otimes_\pi \cdots \otimes_\pi F_m \to F$$

and the functional

$$\tilde{\overline{B}} : F_1 \otimes_\pi \cdots \otimes_\pi F_m \otimes_\pi F^* \to \mathbb{K},$$

both defined in the usual way.

Let H_i $(2 \leqslant i \leqslant m)$ be the identity map on $E_i \otimes_\epsilon E_{i+1} \otimes_\epsilon \cdots \otimes_\epsilon E_m \otimes_\epsilon F^*$, and let J_i $(1 \leqslant i \leqslant m-1)$ be the identity map on $F_1 \otimes_\pi \cdots \otimes_\pi F_i$. Since A_i is integral $(1 \leqslant i \leqslant m)$, the linear map

$$\begin{aligned} A_i \otimes H_{i+1} : {} & E_i \otimes_\epsilon (E_{i+1} \otimes_\epsilon \cdots \otimes_\epsilon E_m \otimes_\epsilon F^*) \\ & \to F_i \otimes_\pi (E_{i+1} \otimes_\epsilon \cdots \otimes_\epsilon E_m \otimes_\epsilon F^*) \end{aligned}$$

is continuous for $1 \leqslant i \leqslant m-1$, and $A_m \otimes \mathrm{Id}_{F^*} : E_m \otimes_\epsilon F^* \to F_m \otimes_\pi F^*$ is also continuous. Consider the operators

$$\begin{aligned} J_i \otimes_\pi (A_{i+1} \otimes H_{i+2}) : {} & F_1 \otimes_\pi \cdots \otimes_\pi F_i \otimes_\pi (E_{i+1} \otimes_\epsilon \cdots \otimes_\epsilon E_m \otimes_\epsilon F^*) \\ & \to F_1 \otimes_\pi \cdots \otimes_\pi F_{i+1} \otimes_\pi (E_{i+2} \otimes_\epsilon \cdots \otimes_\epsilon E_m \otimes_\epsilon F^*). \end{aligned}$$

We shall prove that the composition (which is also continuous)

$$\Psi := \tilde{\overline{B}} \circ \left[J_{m-1} \otimes_\pi (A_m \otimes \mathrm{Id}_{F^*})\right] \circ \cdots \circ \left[J_1 \otimes_\pi (A_2 \otimes H_3)\right] \circ (A_1 \otimes H_2),$$

defined on $E_1 \otimes_\epsilon \cdots \otimes_\epsilon E_m \otimes_\epsilon F^*$ with values in $\mathbb{K}$, coincides with $(\overline{B(A_1,\dots,A_m)})^{\sim}$. Indeed, given $x_{ki} \in E_k$, $f_i^* \in F^*$ $(1 \leqslant k \leqslant m, 1 \leqslant i \leqslant n)$, we have

$$\Psi\left(\sum_{i=1}^n x_{1i} \otimes \cdots \otimes x_{mi} \otimes f_i^*\right) = \tilde{\overline{B}}\left[\sum_{i=1}^n A_1(x_{1i}) \otimes \cdots \otimes A_m(x_{mi}) \otimes f_i^*\right].$$

On the other hand,

$$\begin{aligned}
&\big(\overline{B(A_1,\dots,A_m)}\big)^{\sim}\left(\sum_{i=1}^n x_{1i} \otimes \cdots \otimes x_{mi} \otimes f_i^*\right)\\
&\quad = \sum_{i=1}^n \big(\overline{B(A_1,\dots,A_m)}\big)^{\sim}\big(x_{1i} \otimes \cdots \otimes x_{mi} \otimes f_i^*\big)\\
&\quad = \sum_{i=1}^n \big\langle B(A_1,\dots,A_m)(x_{1i},\dots,x_{mi}), f_i^*\big\rangle\\
&\quad = \sum_{i=1}^n \big\langle B\big(A_1(x_{1i}),\dots,A_m(x_{mi})\big), f_i^*\big\rangle\\
&\quad = \tilde{\overline{B}}\left[\sum_{i=1}^n A_1(x_{1i}) \otimes \cdots \otimes A_m(x_{mi}) \otimes f_i^*\right],
\end{aligned}$$

and this proves the claim.

It remains to show that $\overline{B(A_1,\dots,A_m)}$ is continuous on $E_1 \otimes \cdots \otimes E_m$ endowed with the ϵ-norm. It is enough to check that it coincides with the operator

$$\overline{B} \circ (J_{m-1} \otimes_\pi A_m) \circ \cdots \circ \big[J_1 \otimes_\pi (A_2 \otimes I_3)\big] \circ (A_1 \otimes I_2),$$

where I_i $(2 \leqslant i \leqslant m)$ is the identity map on $E_i \otimes_\epsilon E_{i+1} \otimes_\epsilon \cdots \otimes_\epsilon E_m$. □

Corollary 2.7. *Given Banach spaces E, F and G, a polynomial $Q \in \mathcal{P}(^m G, F)$ and an operator $T \in \mathcal{I}(E, G)$, we have that the composition $Q \circ T \in \mathcal{P}(^m E, F)$ is integral.*

Proof. It is enough to apply Propositions 2.4 and 2.6. □

3. Main result

In this section, we prove that, given $m \in \mathbb{N}$, E is an $\mathcal{L}_\infty$-space if and only if every m-homogeneous 1-dominated polynomial on E is integral. We start by a preparatory result.

Proposition 3.1. *Given $m \in \mathbb{N}$, let E, F be Banach spaces such that $\mathcal{P}_{\mathrm{as}}({}^mE, F) \subseteq \mathcal{P}_{\mathrm{I}}({}^mE, F)$. Then $\mathcal{AS}(E, F) = \mathcal{I}(E, F)$.*

Proof. Suppose that $\mathcal{P}_{\mathrm{as}}({}^mE, F) \subseteq \mathcal{P}_{\mathrm{I}}({}^mE, F)$. Since the inclusion $\mathcal{I}(E, F) \subseteq \mathcal{AS}(E, F)$ is always true, we only have to prove that $\mathcal{AS}(E, F) \subseteq \mathcal{I}(E, F)$.

Fix $e \in E$ and $\phi \in E^*$ with $\|\phi\| \leqslant 1$ so that $\phi(e) = 1$. Define an operator

$$\pi_i : \bigotimes\nolimits_{\pi,s}^{i+1} E \to \bigotimes\nolimits_{\pi,s}^{i} E \quad (1 \leqslant i \leqslant m-1)$$

by

$$\pi_i\left(\sum_{j=1}^{r} \lambda_j x_j \otimes \overset{(i+1)}{\cdots} \otimes x_j\right) = \sum_{j=1}^{r} \lambda_j \phi(x_j) x_j \otimes \overset{(i)}{\cdots} \otimes x_j$$

for all $\lambda_j \in \mathbb{K}$, $x_j \in E$ $(1 \leqslant j \leqslant r)$. As it has been shown in the proof of Theorem 3 in [1], there are operators

$$j_i : \bigotimes\nolimits_{\pi,s}^{i} E \to \bigotimes\nolimits_{\pi,s}^{i+1} E \quad (1 \leqslant i \leqslant m-1)$$

defined in terms of ϕ and e, such that $\pi_i \circ j_i$ is the identity map on $\bigotimes_{\pi,s}^{i} E$.

Choose $T \in \mathcal{AS}(E, F)$, and let $\delta_m : E \to \bigotimes_{\pi,s}^{m} E$ be the m-homogeneous polynomial given by

$$\delta_m(x) = x \otimes \overset{(m)}{\cdots} \otimes x.$$

We show that the polynomial

$$P := T \circ \pi_1 \circ \cdots \circ \pi_{m-1} \circ \delta_m : E \to F$$

is 1-dominated. We proceed by induction on m. For $m = 1$ there is nothing to prove. Suppose that

$$T \circ \pi_1 \circ \cdots \circ \pi_{m-2} \circ \delta_{m-1} : E \to F$$

is 1-dominated. Fix $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$. Then

$$\begin{aligned}
&\sum_{i=1}^{n} \big\| T \circ \pi_1 \circ \cdots \circ \pi_{m-1} \circ \delta_m(x_i) \big\|^{1/m} \\
&\quad = \sum_{i=1}^{n} \big\| T \circ \pi_1 \circ \cdots \circ \pi_{m-1}\big(x_i \otimes \overset{(m)}{\cdots} \otimes x_i\big) \big\|^{1/m} \\
&\quad = \sum_{i=1}^{n} \big\| T \circ \pi_1 \circ \cdots \circ \pi_{m-2}\big(\phi(x_i) x_i \otimes \overset{(m-1)}{\cdots} \otimes x_i\big) \big\|^{1/m} \\
&\quad = \sum_{i=1}^{n} \big|\phi(x_i)\big|^{1/m} \big\| T \circ \pi_1 \circ \cdots \circ \pi_{m-2}\big(x_i \otimes \overset{(m-1)}{\cdots} \otimes x_i\big) \big\|^{1/m}
\end{aligned}$$

$$\begin{aligned}
&\leqslant \left(\sum_{i=1}^{n}\left|\phi(x_i)\right|\right)^{1/m} \\
&\quad \times \left(\sum_{i=1}^{n}\left\|T\circ\pi_1\circ\cdots\circ\pi_{m-2}\circ\delta_{m-1}(x_i)\right\|^{1/(m-1)}\right)^{(m-1)/m} \\
&\leqslant \left(\sum_{i=1}^{n}\left|\phi(x_i)\right|\right)^{1/m}\left(C^{1/(m-1)}\sup_{x^*\in B_{E^*}}\sum_{i=1}^{n}\left|x^*(x_i)\right|\right)^{(m-1)/m} \\
&\leqslant C^{1/m}\sup_{x^*\in B_{E^*}}\sum_{i=1}^{n}\left|x^*(x_i)\right|,
\end{aligned}$$

where we have applied Hölder's inequality, the induction hypothesis and the fact that $\|\phi\|\leqslant 1$. Therefore, P is 1-dominated and hence integral.

Thanks to Proposition 2.5, $\overline{P}:\bigotimes_{\epsilon,s}^m E\to F$ is integral. If $i:\bigotimes_{\pi,s}^m E\to\bigotimes_{\epsilon,s}^m E$ is the natural inclusion, we have that $\overline{P}\circ i:\bigotimes_{\pi,s}^m E\to F$ is integral. We show that $T\circ\pi_1\circ\cdots\circ\pi_{m-1}=\overline{P}\circ i$.

Take $u\in\bigotimes_s^m E$, with

$$u=\sum_{j=1}^{n}\lambda_j x_j\otimes\overset{(m)}{\cdots}\otimes x_j.$$

We have

$$\begin{aligned}
\overline{P}(u) &= \overline{P}\left(\sum_{j=1}^{n}\lambda_j x_j\otimes\overset{(m)}{\cdots}\otimes x_j\right)=\sum_{j=1}^{n}\lambda_j P(x_j) \\
&= \sum_{j=1}^{n}\lambda_j T\circ\pi_1\circ\cdots\circ\pi_{m-1}\left(x_j\otimes\overset{(m)}{\cdots}\otimes x_j\right) \\
&= T\circ\pi_1\circ\cdots\circ\pi_{m-1}\left(\sum_{j=1}^{n}\lambda_j x_j\otimes\overset{(m)}{\cdots}\otimes x_j\right) \\
&= T\circ\pi_1\circ\cdots\circ\pi_{m-1}(u).
\end{aligned}$$

So, $T\circ\pi_1\circ\cdots\circ\pi_{m-1}$ is integral. Therefore,

$$T=T\circ\pi_1\circ\cdots\circ\pi_{m-1}\circ j_{m-1}\circ\cdots\circ j_1:E\to F$$

is integral as well. □

We are now ready to state our main result.

Theorem 3.2. *Let E be a Banach space. The following assertions are equivalent*:

(a) *E is an $\mathcal{L}_\infty$-space*;

(b) *for all $m \in \mathbb{N}$ and every Banach space F, we have $\mathcal{P}_{\mathrm{as}}(^mE, F) \subseteq \mathcal{P}_{\mathrm{I}}(^mE, F)$*;

(c) *there is $m \in \mathbb{N}$ such that for every Banach space F, we have $\mathcal{P}_{\mathrm{as}}(^mE, F) \subseteq \mathcal{P}_{\mathrm{I}}(^mE, F)$.*

Proof. (a) $\Rightarrow$ (b). Let $P \in \mathcal{P}_{\mathrm{as}}(^mE, F)$. By [13, Proposition 3.6] there are an absolutely summing operator $T : E \to G$ and a polynomial $Q \in \mathcal{P}(^mG, F)$ for some Banach space G such that $P = Q \circ T$. Since E is an $\mathcal{L}_\infty$-space, the result of [15] mentioned at the beginning of Section 1 implies that T is integral. By Corollary 2.7, P is integral.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a). It is enough to apply Proposition 3.1 and the result of [15]. □

References

[1] F. Blasco, Complementation in spaces of symmetric tensor products and polynomials, Studia Math. 123 (1997) 165–173.
[2] A. Defant, K. Floret, Tensor Norms and Operator Ideals, in: Math. Studies, Vol. 176, North-Holland, Amsterdam, 1993.
[3] J. Diestel, H. Jarchow, A. Tonge, Absolutely Summing Operators, in: Cambridge Stud. Adv. Math., Vol. 43, Cambridge University Press, Cambridge, 1995.
[4] J. Diestel, J.J. Uhl Jr., Vector Measures, in: Math. Surveys Monographs, Vol. 15, American Mathematical Society, Providence, RI, 1977.
[5] S. Dineen, Complex Analysis on Infinite Dimensional Spaces, in: Springer Monographs in Math., Springer, Berlin, 1999.
[6] K. Floret, Natural norms on symmetric tensor products of normed spaces, Note Mat. 17 (1997) 153–188.
[7] S. Geiss, Ideale multilinearer Abbildungen, Diplomarbeit, Jena (1984).
[8] J. Lindenstrauss, A. Pełczyński, Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications, Studia Math. 29 (1968) 275–326.
[9] M.C. Matos, Absolutely summing holomorphic mappings, An. Acad. Brasil. Ciênc. 68 (1996) 1–13.
[10] Y. Meléndez, A. Tonge, Polynomials and the Pietsch domination theorem, Math. Proc. Roy. Irish Acad. A 99 (1999) 195–212.
[11] J. Mujica, Complex Analysis in Banach Spaces, in: Math. Studies, Vol. 120, North-Holland, Amsterdam, 1986.
[12] A. Pietsch, Ideals of multilinear functionals (designs of a theory), in: H. Baumgärtel, et al. (Eds.), Proceedings of the Second International Conference on Operator Algebras, Ideals, and Their Applications in Theoretical Physics (Leipzig, 1983), in: Teubner-Texte Math., Vol. 67, Teubner, Leipzig, 1984, pp. 185–199.
[13] B. Schneider, On absolutely p-summing and related multilinear mappings, Brandenburgische Landeshochschule Wiss. Z. 35 (1991) 105–117.
[14] C. Stegall, Characterizations of Banach spaces whose duals are L_1 spaces, Israel J. Math. 11 (1972) 299–308.
[15] C.P. Stegall, J.R. Retherford, Fully nuclear and completely nuclear operators with applications to $\mathcal{L}_1$- and $\mathcal{L}_\infty$-spaces, Trans. Amer. Math. Soc. 163 (1972) 457–492.
[16] I. Villanueva, Integral mappings between Banach spaces, J. Math. Anal. Appl., to appear.